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Separating and quasi hyperrigid operator systems in C^*-algebras

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Abstract

In this note, a characterization for boundary representations of a C^*-algebra for an operator system in terms of quasi hyperrigidity and separating property of the operator system is established.

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1 Introduction

In approximation theory, positive approximation processes play an important role. In 1953, P.P Korovkin discovered a simple but powerful method to determine whether a given sequence of positive linear operators on $C[0,1]$ is an approximation process or not. In fact Korovkin proved that it is enough to check the convergence of the sequence on a subset (which later came to be known as Korovkin set) for the purpose. Subsequently, Korovkin's theorem was extended to other function spaces and to abstract spaces such as Banach spaces. As a result a new theory emerged as the Korovkin type approximation theory. In this context Saskin [8] obtained a remarkable result: If X is a compact Hausdorff space and if M is a linear subspace of $C(X)$ containing identity and separating points of X, then M is a Korovkin set if and only if the Choquet boundary for M is the whole of X.

It was William Arveson [3] who initiated the Korovkin type theory in the non-commutative setting. Analogue of Saskin's theorem in the non-commutative setting is proved only for particular cases([3],[6]) where the notion of hyperrigidity captures the essence of Korovkin sets in the context of non commutative C^*-algebras and operator systems. The recent important work of Mathew Kennedy and K R Davidson ([5]) where they essentially show that every operator system and every unital operator algebra has sufficiently many boundary representations to completely norm it is relevant here. Our result, with a weaker notion of hyperrigidity is in similar line of investigation. The notion of separating subalgebra of a C^*-algebra is known. Closely related to this in the context of algebra-subalgebra pair of C^*-algebras are the concepts of *subrepresentations*, *finite representations* and *semi-invariant* subspaces. Arveson gave a characterization of boundary representations for a subalgebra of a C^*-algebra in terms of finiteness of subrepresentations and separating property of the subalgebra [1]. Semi-invariant subspaces and pure completely positive maps are also relevant for our purpose. With a view of extending these results to operator system and the generated C^*-algebra pair, we adapt these notions to this context.

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2 Preliminaries

In this section we recall certain notions and results due to Arveson in non commutative approximation theory relevant to our discussion. We are particularly interested in Arveson's characterization of boundary representations for subalgebras of C^*-algebras in terms of finiteness of representations and separating property of subalgebras.

A closed subspace M of a Hilbert space H is said to be *semi-invariant* under a subalgebra A of $B(H)$-the space of all bounded linear maps on the Hilbert space H, if the map $\varphi(T) = P_M T|_M$ is multiplicative on A. This definition is due to Sarason [7], who also proved the following characterization of semi-invariant subspaces. If M is a semi-invariant subspace for an algebra A, then $M_0 = [AM] \ominus M$ is A-invariant, so that $M = [AM] \ominus M_0$ is a nested difference of A-invariant subspaces where $[AM]$ is the norm closure of $\{a\xi; a \in A \text{ and } \xi \in H\}$ in H. Conversely if $N = M_1 \ominus M_0$ where $M_0 \subseteq M_1$ are A-invariant, then N is semi-invariant for A. Note that when A is self-adjoint, semi-invariant subspaces are reducing. But in general semi-invariant subspaces need not be even invariant.

If A is a subalgebra of a C^*-algebra $\mathcal{B}$ containing the identity of $\mathcal{B}$, then a representation of A on a Hilbert space H is a homomorphism φ from A into the algebra of operators $B(H)$ satisfying the conditions: (i) $\varphi(e) = I$ and (ii) $\|\varphi(a)\| \leq \|a\|$ for all $a \in A$. Note that when $A = \mathcal{B}$, φ will become the usual representation of the C^*-algebra. If M is a semi-invariant subspace for $\varphi(A)$, then the new representation φ_0 of A on M defined by $\varphi_0(a) = P_M\varphi(a)|_M,\ a \in A$ is called a subrepresentation of φ. Here again, when A is self-adjoint, φ_0 will become the usual subrepresentation.

Let A be a subalgebra of a C^*-algebra and let φ be a representation of A on some Hilbert space H. Two representations φ_1 and φ_2 of A on a Hilbert space H are said to be *(unitarily) equivalent* if there exists a unitary $U \in B(H)$ such that $\varphi_1(a) = U^*\varphi_2(a)U$ for all $a \in A$. The representation φ is called *infinite* if it is equivalent to a proper subrepresentation $\varphi_0 \neq \varphi$; φ is called *finite* if it is not infinite. Arveson proved the following characterization of finite representations: φ is finite if and only if for every isometry $V \in B(H)$, the condition $V^*\varphi(a)V = \varphi(a)$ for all $a \in A$ implies V is unitary. We will use this as the definition of finite representation of a subalgebra.

Definition 2.1. An *operator system* S in a C^*-algebra $\mathcal{A}$ is a self-adjoint linear subspace of $\mathcal{A}$ containing the identity of $\mathcal{A}$ such that $\mathcal{A} = C^*(S)$, the C^*-algebra generated by S.

The set of all equivalence classes of irreducible representations of a C^*algebra $\mathcal{A}$ is called the spectrum of $\mathcal{A}$ and is denoted by $\hat{\mathcal{A}}$. Consider an operator system S and a Hilbert space H. Let $CP(S, B(H))$ denote the set of all completely positive (CP) maps from S to $B(H)$ and let $UCP(S, B(H))$ be the set of all CP maps that are unital. A map $\varphi \in UCP(S, B(H))$ is called pure if whenever $\varphi - \xi$ is CP for some $\xi \in CP(S, B(H))$ then there exists $0 \leq t \leq 1$ such that $\xi = t\varphi$.

Definition 2.2. Let S be an operator system in a C^*-algebra $\mathcal{A}$. A *boundary representation* of $\mathcal{A}$ for S is an irreducible representation π of $\mathcal{A}$ on a Hilbert space such that $\pi_{|S}$ has a unique completely positive extension, namely π itself to $\mathcal{A}$.

Definition 2.3. A set G (finite or countably infinite) of generators of an abstract C^*-algebra $\mathcal{A}$ is said to be *hyperrigid* if for every faithful representation $\mathcal{A} \subseteq B(H)$ of $\mathcal{A}$ on a Hilbert space and every sequence of unital completely positive maps $\{\varphi_n\}$ from $B(H)$ to itself,

$$\lim_{n\to\infty} \|\varphi_n(g) - g\| = 0, \quad \forall\ g \in G \Rightarrow \lim_{n\to\infty} \|\varphi_n(a) - a\| = 0, \quad \forall\ a \in \mathcal{A}.$$

3 Quasi hyperrigidity and separating operator systems

The notion of hyperrigidity for operator system introduced by Arveson is closely related to the theory of extreme points and Choquet boundary of operator systems. Arveson proved that when the operator system is hyperrigid, Choquet boundary of the operator system is equal to the spectrum of the generated C^*-algebra. Weaker analogues of the notion of hyperrigidity also proved to be worth studying in exploring the related concepts. In this context, the notion of quasi hyperrigidity for operator system was first introduced in [9]. We give below the definition of a quasi hyperrigid operator system and give an example to show that the notion is weaker than the notion of hyperrigidity.

Definition 3.1. An operator system S is said to be *quasi hyperrigid* if for every irreducible representation π of $C^*(S)$ and for every isometry $V : H_\pi \to H_\pi$ such that $V^*\pi(s)V = \pi(s)$ for all s in S, then $V^*\pi(a)V = \pi(a)$ for all a in $C^*(S)$.

Example 3.2. This example is taken from a previous paper [9] co-authored by the authors. Let $M_n(\mathbb{C})$ denote the set of all $n \times n$ matrices over $\mathbb{C}$, where $n \geq 3$. Define a unital completely positive map Φ on $M_n(\mathbb{C})$ as given below. Let

$$M = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots\ldots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \ldots\ldots & a_{3n} \\ . & . & . & \ldots\ldots & . \\ . & . & . & \ldots\ldots & . \\ a_{n1} & a_{n2} & a_{n3} & \ldots\ldots & a_{nn} \end{bmatrix}$$

be arbitrary. Now define Φ on $M_n(\mathbb{C})$ as $\Phi(M) = N$, where

$$N = \begin{bmatrix} a_{11} & a_{12} & 0 & \ldots\ldots & 0 \\ a_{21} & a_{22} & 0 & \ldots\ldots & 0 \\ 0 & 0 & a_{22} & \ldots\ldots & 0 \\ 0 & 0 & 0 & a_{22} & 0 \\ . & . & . & \ldots\ldots & . \\ 0 & 0 & 0 & \ldots\ldots & a_{22} \end{bmatrix}.$$

Now let $M = T$, where $a_{21} = 1$ and all other entries equal to 0. If $S = span\{I, T, T^*\}$ and $A = C^*(S)$, then $\Phi(s) = s$ for all s in S, but $\Phi(TT^*) \neq TT^*$. i.e, S is not a hyperrigid set. However, if V is any isometry such that $V^*V = I$, then $VV^* = I$, since A is finite dimensional. Thus S is quasi hyperrigid, but fails to be a hyperrigid set.

More examples of quasi hyperrigid systems are given in [9].

The following proposition is on a relation between finite representation of an algebra and quasi hyperrigidity of the associated operator system in the generated C^*-algebra.

Proposition 3.3. Let A be a unital operator algebra in $B(H)$. Consider the operator system $S = A + A^*$and let $\mathcal{B} = C^*(S)$ be the C^*-algebra generated by S. Then every irreducible representation $\pi \in \widehat{\mathcal{B}}$ is a finite representation of A if and only if S is a quasi hyperrigid system.

Proof. Assume that every irreducible representation $\pi \in \widehat{\mathcal{B}}$ is a finite representation of A. Let $\pi : \mathcal{B} \longrightarrow B(H_\pi)$ be an irreducible representation and let $V : H_\pi \longrightarrow H_\pi$ be an isometry satisfying

$$V^*\pi(s)V = \pi(s) \ \text{ for all } \ s \in S.$$

.

In particular, $V^*\pi(a)V = \pi(a)$ for all $a \in A$.

But by our assumption π is a finite representation of A and therefore V is unitary. Hence we have

$$V^*\pi(s)V = \pi(s) \ \text{ for all } \ s \in \mathcal{B}.$$

This will imply that S quasi hyperrigid.

Conversely, assume that S is a quasi hyperrigid system. Let $\pi : \mathcal{B} \longrightarrow B(H_\pi)$ be an irreducible representation and let $V : H_\pi \longrightarrow H_\pi$ be an isometry satisfying

$$V^*\pi(a)V = \pi(a) \ \text{ for all } \ a \in A.$$

$$V^*\pi(s)V = \pi(s) \ \text{ for all } \ s \in S = A + A^*.$$

Since S is quasi hyperrigid, we have

$$V^*\pi(s)V = \pi(s) \ \text{ for all } \ s \in \mathcal{B}.$$

V is an isometry and from the above equation it is clear that the range of V reduces the irreducible C^*-algebra $\pi(\mathcal{B})$. Therefore V is unitary and hence π is a finite representation of A. Q.E.D.

Now we define a separating operator system in line with the definition of a separating subalgebra defined by Arveson (Def. 2.4.4., [1]). The necessary and sufficient condition due to Arveson for a subalgebra to separate an irreducible representation is as follows. A subalgebra A of a C^*-algebra $\mathcal{B}$ separates an irreducible representation ω of $\mathcal{B}$ on a Hilbert space H if and only if the following condition is satisfied: for every irreducible representation π of $\mathcal{B}$ on a Hilbert space K and every isometry $V \in L(H,K), V^*\pi(a)V = \omega(a)$ for all $a \in A$ implies that π and ω are unitarily equivalent representations of $\mathcal{B}$. Replacing subalgebra with a linear subspace we give the definition of a separating operator system as follows.

Definition 3.4. Let S be an operator system and $\mathcal{A} = C^*(S)$. Let $\pi : \mathcal{A} \longrightarrow B(H)$ be an irreducible representation. We say that S separates π if for every irreducible representation $\rho : \mathcal{A} \longrightarrow B(K)$ and every isometry $V : H \longrightarrow K$, $V^*\rho(s)V = \pi(s)$, for all $s \in S$ implies that π and ρ are unitarily equivalent representations of $\mathcal{A}$. S is called a *separating operator system of A if it separates every irreducible representation of $\mathcal{A}$.*

In the classical case, a set S in $C(X)$ (where X is compact Hausdorff) is said to separate points of X if for each pair of points $x_1, x_2 \in X$ where $x_1 \neq x_2$, there exists $g \in S$ such that $g(x_1) \neq g(x_2)$. As irreducible representations of $C(X)$ correspond to points of X, our notion of separating operator system will coincide with the subspace which separates points in the classical sense. Further, when S is a Korovkin set in $C(X)$, it separates points of X([4], page 163). In the same way, in non commutative setting, if S is a separable operator system and $\mathcal{A} = C^*(S)$, then by ([3], Theorem 2.1) it follows that every separable hyperrigid operator system is separating.

Now we will establish our main result.

Theorem 3.5. Let S be an operator system and $\mathcal{A} = C^*(S)$. Then every irreducible representation of $\mathcal{A}$ is a boundary representation of $\mathcal{A}$ for S if and only if the following conditions are satisfied:

(i) S is quasi hyperrigid;

(ii) every irreducible representation of $\mathcal{A}$ restricted to S is pure;

(iii) S is a separating operator system.

Proof. Assume that every irreducible representation for S is a boundary representation. Consider an irreducible representation $\pi : \mathcal{A} \longrightarrow B(H_\pi)$. Let $V : H_\pi \longrightarrow H_\pi$ be an isometry such that $V^*\pi(s)V = \pi(s)$ for every $s \in S$. Then $V^*\pi(.)V$ is a completely positive map on $\mathcal{A}$ which agrees with π on S. But π is a boundary representation of $\mathcal{A}$ for S. This implies that $V^*\pi(a)V = \pi(a)$ for all $a \in \mathcal{A}$. Therefore S is quasi hyperrigid.

Let $\pi_{|_S} = \varphi_1 + \varphi_2$ for some $\varphi_i \in CP(S, B(H_\pi)), i = 1, 2$. Then by ([1],Theorem 1.2.3), there exists $\xi_i \in CP(\mathcal{A}, B(H_\pi))$ such that $\xi_{i|_S} = \varphi_i, i = 1, 2$. Then $\xi_1 + \xi_2$ is a completely positive extension of $\pi_{|_S}$. But π is a boundary representation. Then $\pi = \xi_1 + \xi_2$. Since π is an irreducible representation, π is pure([1],Theorem 1.4.3). Therefore there exists $t_i \geq 0, i = 1, 2$ such that $\xi_i = t_i\pi, i = 1, 2$. This implies that $\varphi_i = t_i\pi_{|_S}, i = 1, 2$ and therefore $\pi_{|_S}$ is pure.

Now we will prove that S is a separating operator system by showing that S separates π. Let $\rho : \mathcal{A} \longrightarrow B(H_\rho)$ be any other irreducible representation and let $V : H_\pi \longrightarrow H_\rho$ be an isometry satisfying the condition $V^*\rho(s)V = \pi(s)$, for all $s \in S$. We know that $V^*\rho(.)V$ is a completely positive extension of $\pi_{|_S}$ and since π is a boundary representation of $\mathcal{A}$ for S, we have $V^*\rho(a)V = \pi(a)$, for all $a \in \mathcal{A}$. But then VH_π is a reducing subspace for $\rho(\mathcal{A})$. Since $\rho(\mathcal{A})$ is irreducible, we must have $VH_\pi = H_\rho$ and this gives that V is unitary. Therefore π and ρ are unitarily equivalent. Since π is arbitrary, we get that S is a separating operator system.

Conversely, assume that conditions (i), (ii) and (iii) are satisfied. Let π be an irreducible representation of $\mathcal{A}$ on a Hilbert space H_π. In order to prove that π is a boundary representation of $\mathcal{A}$ for S. Consider

$$K = \{\xi \in CP(\mathcal{A}, B(H_\pi)) : \xi_{|_S} = \pi_{|_S}\}.$$

We will show that $K = \{\pi\}$. The space $CP(\mathcal{A}, B(H_\pi))$ has a natural topology called BW-topology defined by Arveson ([1], page 146) where a net of maps in it converges to a limit if the image net of operators in $B(H)$ converges to the corresponding limit with respect to the weak operator topology for every element of $\mathcal{A}$. With respect to BW-topology, K is a compact convex subset of $CP(\mathcal{A}, B(H_\pi))$([1], page 146). Obviously K is non-empty. By Krein-Milman theorem, K is the closed convex hull of its extreme points. Let $\varphi \in K$ is an extreme point. We will show that $\varphi = \pi$.

We first claim that φ is a pure element of $CP(\mathcal{A}, B(H_\pi))$. Choose non-zero elements φ_1 and φ_2 of $CP(\mathcal{A}, B(H_\pi))$ such that $\varphi(a) = \varphi_1(a) + \varphi_2(a), a \in \mathcal{A}$. Then π and φ are bounded linear maps of $\mathcal{A}$ agreeing on S. But by condition (2) of the theorem there exist scalars $t_i \geq 0, i = 1, 2$ such that $\varphi_i(s) = t_i\pi(s)$, for every $s \in S$. If we take $t_i = 0$, and since $e \in S$, we get $\varphi_i(e) = 0$. Hence $\varphi_i = 0, i = 1, 2$ which is not possible because of our selection of φ_i. This gives that $t_i > 0, i = 1, 2$. Since $e \in S, \pi(e) = 1 = t_1\pi(e) + t_2\pi(e)$ we get $t_1 + t_2 = 1$. Now put $\psi_i = t_i^{-1}\varphi_i$.

Then $\psi_i \in K, i = 1, 2$. Therefore we get $\varphi = t_1\psi_1 + t_2\psi_2$. But by our assumption, φ is an extreme point of K, $\varphi = \psi_1 = \psi_2$. Then $\varphi_i = t_i\varphi, i = 1, 2$. This proves that φ is pure.

By ([1], Theorem 1.4.3), there exists an irreducible representation $\rho : \mathcal{A} \longrightarrow B(H_\rho)$ and a bounded operator $V : H_\pi \longrightarrow H_\rho$ such that $\varphi(a) = V^*\rho(a)V$ for all $a \in \mathcal{A}$. Then $\pi(s) = V^*\rho(s)V$ for all $s \in S$. Putting $s = e$ we get $V^*V = I$ and hence V is an isometry. Because of our assumption that S is a separating operator system, we get that π is unitarily equivalen to ρ. Therefore, there exists a unitary operator $U : H_\rho \longrightarrow H_\pi$ such that $\rho = U^{-1}\pi U$. Hence we can write $\pi(s) = (UV)^*\pi(s)(UV)$ for all $s \in S$. But UV is an isometry. By our assumption (i) S is quasi hyperrigid and this implies that UV is unitary which in turn gives $V = U^{-1}UV$ is unitary. Therefore, we can write $\pi(s) = V^{-1}\rho(s)V, s \in S$. Then $V^{-1}\rho(.)V$ is a representation of $\mathcal{A}$ which agrees with π on S. This gives that $\pi(a) = V^{-1}\rho(a)V, a \in \mathcal{A}$. Therefore we have $\pi = \varphi$ and the proof is complete. Q.E.D.

In the classical case the above theorem can be viewed as a restatement of the known fact that Korovkin sets are separating.

The following example illustrates the above theorem.

Example 3.6. Let $G = span\,(I, S, S^*, SS^*)$, where S is the unilateral right shift in $B(\mathcal{H})$ and I the identity operator. Let $A = C^*(G)$ be the C^*-algebra generated by G. We have, $K(\mathcal{H}) \subseteq A$. $A/K(\mathcal{H}) \cong C(\mathbb{T})$ is commutative, where $\mathbb{T}$ denotes the unit circle in $\mathbb{C}$ and the spectrum $\hat{A}$ of A can be identified with $\{Id\} \cup \mathbb{T}$. Since S is an isometry, G is hyperrigid ([3], Theorem 3.3) and this will imply that all the irreducible representations of A are boundary representations for S. Clearly G is quasi hyperrigid. Also S is separating operator system. Further, $Id_{|_G}$ is pure and the irreducible representations corresponding to $\mathbb{T}$ are one dimensional and their restrictions to S are also pure.

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